On search for the M-Theory Lagrangian

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Abstract

We present a starting point for the search for a Lagrangian density for M-Theory using characteristic classes for flat foliations of bundles. PACS classification: 11.10.-z, 11.15.-q, 11.30.Ly

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The starting point of our investigation is [1]. In this paper a non-abelian generalisation of antisymmetric rank 2 tensor gauge potential—denoted B— theory was formulated. The key point of that discussion was the appearence of a *flat* connection 1-form which was defined via B. The first task was to try to understand *why* one actually needs a flat connection 1-form to play such a dominant role in that theory. The reason according to our understanding is as follows: it is a well-known fact that connections in geometry are used to define the notion of a *parallel* propagation or *covariant differentiation*. (The mathematical setting is that of a principal bundle P over a base space M with structure group G). The parallel propagation refers to sections of some associated vector bundle E say, to P. But there is one step that comes *before* that; namely, in order to define this parallel propagation one has to define the notion of a *horizontal lift* of a *curve* on the base space to a curve on the total space (here the word "horizontal" means that the vector field tangent to the lifted curve is horizontal). This is exactly the reason why the connection is a *1-form*, namely it is the Poincare dual of a curve which is a 1-chain.

It is then clear that if we are to introduce *2-forms* as gauge potentials and if the recepie that gauge potentials correspond to a way of defining parallel propagation holds, then one can see that in this case 2-forms would correspond to parallel propagation of sections of an associated vector bundle *along a surface*. This is indeed the case in string theory for instance where the surface is the worlsheet replacing the notion of paths, curves, of point particles. In principal bundle language this means that we would like to have *horizontal lifts of surfaces* (a surface is a 2-chain).

Now in the ordinary case of 1-chains we know from elementary theory of ODE's that a horizontal lift of a curve always exists (but it need not be unique). For a 2-chain however, such a statement—i.e. existence of a horizontal lift— does not follow automatically. The reason is that now we are dealing with PDE's because we have to consider a 2-parameter family of transformations on the base manifold, namely the flows of *two* vector fields which are the vector fields which generate the surface we would like to lift horizontally. One then needs an *integrability* condition for this PDE to have a solution (and hence for the horizontal lift to exist). This is due to the fact that the horizontal lifts of each of the vector fields (which generate the

surface we want to lift) definitely *exist* but *they may not form a surface* on the total space. The necessary condition then for this to happen is that the (Lie algebra) commutator of these two horizontal vector fields must also be a horizontal vector field. In general this is not true, unless the connection 1-form we used to lift them separately is *flat*.

So we see why to each gauge potential 2-form there corresponds a fundamental *flat connection 1-form* which is naturally attached to it.

What happens then if we would like to deal with *p-branes*? Then one has to use $(p + 1)$-forms as gauge potentials because we now want to lift horizontally a $(p + 1)$-chain. We know from the theory of integrable systems that the condition of this connection 1-form (used to lift separately each one of the vector fields which generate the $(p + 1)$-chain we want to lift) being flat is enough to guarantee the existence of a solution to the relevant PDE which in turn means that the horizontal lift of this $(p + 1)$-chain exists. (One has to be a little more careful with the holonomy but this does not affect our argument).

We learn therefore from the above discussion that if one wants to keep for higher degree gauge potentials the role that connection 1-forms play in the case of point particles, there must be an underlying *flat* connection 1-form which is the fundamental object. In [1] the relation between this flat connection 1-form and the gauge potential B for the case of the base manifold being 4-dimensional Minkowski space was given explicitly.

This oughts to be true for an additional reason; as pointed out in [1], it is exactly due to the existence of this flat connection 1-form that the interacting B-theory is (classically) equivalent to a non-linear σ model. And we also know that all theories involving extended objects *are*, or can be thought of as, in fact σ models (at least in their bosonic sector if we want to assume supersymmetry). The only difference then with strings is that the source space will be—for a theory of p-branes—a Riemannian $(p + 1)$-dimensional manifold instead of just a Riemann surface. So on top of our previous geometric discussion, flat connection 1-forms are fundamental also for physical reasons: they are reminiscents of the fact that our theory is in fact a σ model. At this point we would also like to recall [2] in which Polyakov rewrites σ model Lagrangians using flat connection 1-forms (if the source space is not

simply-connected one has to be a little more careful as whether all flat connection 1-forms can be obtained in this way but this is not important for our immediate discussion).

We now pass on to M-theory. This is supposed to be a supersymmetric non-perturbative theory on an *11-dimensional* manifold (see [3] for a good introduction). The crucial part is the *soliton* part of this theory since dualities can take care of the rest. Let us assume that we start looking for a suitable Lagrangian density for, at least to begin with, the bosonic sector of this theory. Moreover our discussion will be restricted only to the classical level.

When one seeks for a non-perturbative theory Lagrangian—recall that we are interested in the soliton part of the theory— the most sensible place to start looking at is *characteristic classes* because they are quite "stable" as being topological. But the first difficulty comes forth immediately: we need an 11-form since the space is 11-dimensional for a Lagrangian density and all characteristic classes we know of in usual Chern-Weil theory are of *even* degree. But this is not the case for *characteristic classes for flat foliations of bundles* bearing in mind our previous discussion for flat connection 1-forms as being the really fundamental objects for theories with extended objects. In fact, characteristic classes for flat foliations of bundles exist *only* in *odd dimensional cohomology* [4]. What is even more exciting is that unlike the usual characteristic classes of Chern-Weil theory they are not defined using the curvature of a connection 1-form but *flat connection 1-forms themselves*. These are definitely interesting coincidences, at least for the moment. (*Note:* For *arbitrary* foliations, characteristic classes in *any* dimension may occur; they are "combinations" of ordinary characteristic classes—Chern-Weil theory—and of the characteristic classes for flat foliations of bundles mentioned above; this comment refers to F-theory).

Well, we then just go to the relevant theory (also related to Gelfand-Fuchs cohomology and to graph cohomology—the later in 3-dimensions—) of characteristic classes for foliations, in the special case in hand of flat foliations of bundles and we just pick up the 11th-dimensional class which is just

$$\Delta_*(y_6) \in H_{DR}^{11}(M)$$

where M is our 11-dimensional base manifold, $H_{DR}^{11}(M)$ is its 11th-dimensional

de Rham cohomology group and y_6 is the 6th Samelson generator

$$y_6 := \sigma \tilde{c}_6$$

where $\tilde{c}_6$ is the 6th elementary symmetric function c_6 multiplied by a factor $(i)^6$. The other maps Δ_* and σ appearing in the formula are explained in detail in, for example, [4]. This is moreorless standard material in the geometry of foliations; the map Δ_* is roughly speaking the analogue of the Weil homomorphism and the map σ is the map induced in cohomology level by the universal transgression operator.

The last point we would like to mention is actually *conjectural*: in [5] an invariant was introduced for arbitrary foliations using the *even* pairing between *K-homology* and *cyclic cohomology*. This number then we conjecture that should be very closely related to the *topological charge* for the above mentioned Lagrangian density.

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